Ring Theory 15

06 July 2024 22:00

Q> Prove that if R is an UFD where every maximal ideal is principle then it is a PID.

Ans:— We need to show that every non-zero prime ideal is maximum as in principal ideal domain every prime ideal is maximum.

Let $I \subset R$ be non-zero prime ideal of R

Let $I \subset M$ for M maximal and $M \subseteq R$

$\Rightarrow M = \langle a \rangle$

$xy \in I \Rightarrow x \in I$ or $y \in I$

$\Rightarrow x \in \langle a \rangle$ or $y \in \langle a \rangle$

$\Rightarrow a | x$ or $a | y$ → if x, y be primes in I

$\Rightarrow$ a and x or y are associates

$\Rightarrow \langle x \rangle = \langle a \rangle$ or $\langle y \rangle = \langle a \rangle$

$\Rightarrow I = M$

$\Rightarrow \Leftarrow$

$\Rightarrow I$ is maximal

Q> Prove that 2 is irreducible in $\mathbb{Z}[\sqrt{-7}]$ but 2 is not prime in it.

Ans:— Let $2 = ab$

$N(2) = N(ab) = ab(\overline{ab}) = |a^2||b^2| = 4$

Case I:— $|a^2| = |b^2| = 2$

Let $\exists\, r \in \mathbb{Z}[\sqrt{-7}]$ such that $N(r) = 2 \Rightarrow N(a+\sqrt{-7}b) = 2$

$\Rightarrow a^2 + 7b^2 = 2$

$\Rightarrow \Leftarrow$

Case II:— WLOG $|a^2| = 1$, $|b^2| = 4$

$\Rightarrow a = \pm 1 \Rightarrow b = \pm 2$

$\Rightarrow a$ is a unit

$\Rightarrow 2$ is not reducible

Now we need to show that 2 is not prime.

$$8 = (a+\sqrt{-7}\,b)(a-\sqrt{-7}\,b) = a^2+7b^2 \Rightarrow a=1,\ b=1$$

$2 \mid 8$ but does $2 \mid 1+\sqrt{-7}$ or $2 \mid 1-\sqrt{-7}$?

If $2 \mid 1+\sqrt{-7}$ then $1+\sqrt{-7} = 2(a+b\sqrt{-7}) = 2a+2b\sqrt{-7}$

$\Rightarrow a \notin \mathbb{Z}$

Similarly for $1-\sqrt{-7}$.

$\Rightarrow 2$ is not prime

To Do in Next Class:-

Q) Tell whether $\mathbb{Z}[x,y]$ is a UFD, PID, ED or not?